Sample Paper

Class XII (2017-18)

Mathematics

Time allowed: 3 hrs. **Maximum Marks: 70**

GENERAL INSTRUCTIONS:

(i) **All questions are compulsory**.

(ii) The question paper consists of **29** questions divided into four sections **A, B, C** and **D.**

(iii) Section **A** contains **4** questions of **1** mark each. Section **B** contains **8** questions of **2** marks each. Section **C** contains **11** questions of **4** marks each. Section **D** contains **6** questions of **6** marks each.

(iv) There is no overall choice. However, an internal choice has been provided in 3 questions of 4 marks each and **3** questions of **6** marks each. You have to attempt only one of the alternatives in all such questions.

(v) Use of **calculators** is not permitted.

[SECTION A]

1. Find the values of x, y and z

$$\begin{bmatrix} x+y & 2 \\ 5+z & xy \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 5 & 8 \end{bmatrix}$$

2. Find *gof* and *fog*, if $f(x) = 8x^3$ and $g(x) = x^{\frac{1}{3}}$.

3. Find an anti derivative (or integral) of the functions

$$f(x) = (ax + b)^2$$

4. If a line makes angles 90°, 135°, 45° with the *x*, *y* and *z*-axes respectively, find its direction cosines.

[SECTION B]

5. Find *X* and *Y*, if $X + Y = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$

6. Find the intervals in which the function *f* given by $f(x) = 2x^2 - 3x$ is

(a) Strictly increasing **(b)** Strictly decreasing

7. Find the equation of the normal to the curve $x^2 = 4y$ which passes through the point (1, 2).

8. Find the maximum profit that a company can make, if the profit function is given by

$$p(x) = 41 - 72x - 18x^2$$

9. The Cartesian equation of a line is

$$\frac{x+3}{2}=\frac{y-5}{4}=\frac{z+6}{2}$$

Find the vector equation for the line.

10. A and B are two events such that $P(A)\neq 0$. Find $P(B/A)$, if

(i) A is a subset of B **(ii)** $A\cap B=\phi$

11. Solve the following linear programming problem graphically:

Maximise $Z = 4x + y$ …(1)

Subject to the constraints

$x+y\le 50$ …(2)

$3x+y\le 90$ …(3)

$x\ge 0, y\ge 0$ …(4)

12. Evaluate $\int\frac{dx}{x^2-6x+13}$

[SECTION C]

13. If $0<x<1$, and $\sqrt{1+x^2}\left[\left\{x\cos\left(\cot^{-1}x\right)+\sin\left(\cot^{-1}x\right)\right\}^2-1\right]^{1/2}=2x$ then find the value of x.

14. If $A=\begin{bmatrix}0 & -\tan\frac{\alpha}{2}\\ \tan\frac{\alpha}{2} & 0\end{bmatrix}$ and I is the identity matrix of order 2, show that

$$I+A=(I-A)\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}$$

OR

Prove that $\begin{vmatrix}a & a+b & a+b+c\\ 2a & 3a+2b & 4a+3b+2c\\ 3a & 6a+3b & 10a+6b+3c\end{vmatrix}=a^3.$

15. If $y=\left(\tan^{-1}x\right)^2$, show that $\left(x^2+1\right)^2\frac{d^2y}{dx^2}+2x\left(x^2+1\right)\frac{dy}{dx}=2$

16. Find $\int\frac{(3\sin\phi-2)\cos\phi}{5-\cos^2\phi-4\sin\phi}d\phi$

17. Evaluate $\int_0^{\frac{\pi}{2}}\log\sin x\,dx$

18. **(a)** If $\hat{i}+\hat{j}+\hat{k}$, $2\hat{i}+5\hat{j}$, $3\hat{i}+2\hat{j}-3\hat{k}$ and $\hat{i}-6\hat{j}-\hat{k}$ are the position vectors of points A, B, C and D respectively, then find the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$. Deduce that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear.

(b) Three vectors $\vec{a},\vec{b}$ and $\vec{c}$ satisfy the condition $\vec{a}+\vec{b}+\vec{c}=\vec{0}$. Evaluate the quantity

$$\mu=\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a},\ \text{if } |\vec{a}|=1, |\vec{b}|=4 \text{ and } |\vec{c}|=2.$$

19. Verify that the function $y = c_1 e^{ax} \cos bx + c_2 e^{ax} \sin bx,$ where c_1, c_2 are arbitrary constants is a solution of the differential equation $\frac{d^2 y}{dx^2} - 2a\frac{dy}{dx} + \left(a^2 + b^2\right) y = 0$.

20. If $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular vectors of equal magnitudes, show that the vector $\vec{a} + \vec{b} + \vec{c}$ is equally inclined to $\vec{a}, \vec{b}$ and $\vec{c}$. Also find the angle which $\vec{a} + \vec{b} + \vec{c}$ makes with $\vec{a}$ or $\vec{b}$ or $\vec{c}$.

21. Let *X* denote the number of hours you study during a randomly selected school day. The probability that *X* can take the values *x*, has the following form, where k is some unknown constant.

$$P\left(X = x\right) = \begin{cases} 0.1, \text{ if } x = 0 \\ kx, \text{if } x = 1 \text{ or } 2 \\ k\left(5 - x\right), \text{if } x = 3 \text{ or } 4 \\ 0, \text{ otherwise} \end{cases}$$

(a) Find the value of *k*.

(b) What is the probability that you study at least two hours? Exactly two hours? At most two hours?

22. A doctor is to visit a patient. From the past experience, it is known that the probabilities that he will come by train, bus, scooter or by other means of transport are respectively $\frac{3}{10}, \frac{1}{5}, \frac{1}{10}$ and $\frac{2}{5}$. The probabilities that he will be late are $\frac{1}{4}, \frac{1}{3},$ and $\frac{1}{12},$ if he comes by train, bus and scooter respectively, but if he comes by other means of transport, then he will not be late. When the arrives, he is late. What is the probability that he comes by train?

23. A manufacturer has three machines I, II and III installed in his factory. Machines I and II are capable of being operated for at most 12 hours whereas machine III must be operated for at least 5 hours a day. She produces only two items M and N each requiring the use of all the three machines. The number of hours required for producing 1 unit of each of M and N on the three machines are given in the following table.

Items	Number of hours required on machines		
	I	II	III
M	1	2	1
N	2	1	1.25

She makes a profit of Rs 600 and Rs 400 on items M and N respectively. How many of each item should she produce so as to maximize her profit assuming that she can sell all the items that she produced? What will be the maximum profit?

[SECTION D]

24. The sum of three numbers is 6. If we multiply third number by 3 and add second number to it, we get 11. By adding first and third numbers, we get double of the second number. Represent it algebraically and find the numbers using matrix method.

25. Consider the binary operations * : R × R $\to$ R and o : $R \times R \to R$ defined as $a * b = |a - b|$ and $aob = a, \forall a, b \in R$. Show that * is commutative but not associative, o is associative but not commutative. Further, show that $\forall a, b, c \in R, a*(b o c) = (a*b) o (a*c)$. Does o distribute over * ? Justify your answer.

OR

Consider $f : R^{+} \to (-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$. Show that f is invertible with $f^{-1}(y) = \left(\frac{(\sqrt{y+6}) - 1}{3}\right)$.

Hence find :

(i) $f^{-1}(3)$ **(ii)** y, if $f^{-1}(y) = \frac{4}{3}$

26. Show that height of the cylinder of greatest volume which can be inscribed in a right circular cone of height *h* and semi vertical angle α is one-third that of the cone and the greatest volume of cylinder is $\frac{4}{27}\pi h^3 \tan^2 \alpha$.

27. Find the area lying above *x*-axis and included between the circle $x^2 + y^2 = 8x$ and inside of the parabola $y^2 = 4x$.

28. Show that the differential equation $2ye^{\frac{x}{y}}dx + \left(y - 2xe^{\frac{x}{y}}\right)dy = 0$ is homogenous and find its particular solution, given that, $x = 0$ when $y = 1$.

29. A line makes angles α, β, γ and δ with the diagonals of a cube, prove that $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + \cos^2 \delta = \frac{4}{3}$.

Vidyamandir Classes
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Answers to Sample Paper

Class XII (2017-18)

Mathematics

[SECTION A]

1. Here two matrices are equal so comparing the corresponding elements

$x + y = 6$ ...(1)

$5 + z = 5 \Rightarrow z = 0$

and $x.y = 8$ ...(2)

Solving (i) and (ii) we get : $x = 4$ or 2 and $y = 2$ or 4

2. Given that $f(x) = 8x^3$ and $g(x) = x^{1/3}$

Now $gof = g(f(x)) = g(8x^3) = (8x^3)^{1/3} = 2x$

and $fog = f(g(x)) = f\left(x^{1/3}\right) = 8\left(x^{1/3}\right)^3 = 8x$

3. $\int (ax+b)^2 dx$

$$= \frac{(ax+b)^{2+1}}{2+1} \times \frac{1}{a} \quad \left\{ \because \int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{n+1} \times \frac{1}{a} \right\}$$

$$= \frac{1}{3a}(ax+b)^3 + C$$

4. Here line makes 90°, 135°, 45° with *x*, *y* and *z* axes respectively. So $l = \cos 90°$, $m = \cos 135°$, $n = \cos 45°$, $l = 0, m = \frac{-1}{\sqrt{2}}, n = \frac{1}{\sqrt{2}}$

[SECTION B]

5. We have $(X+Y)+(X-Y) = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$

or $(X+X)+(Y-Y) = \begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix} \Rightarrow 2X = \begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix}$ or $X = \frac{1}{2}\begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}$

Also $(X+Y)-(X-Y) = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$

or $(X-X)+(Y+Y) = \begin{bmatrix} 5-3 & 2-6 \\ 0 & 9+1 \end{bmatrix} \Rightarrow 2Y = \begin{bmatrix} 2 & -4 \\ 0 & 10 \end{bmatrix}$ or $Y = \frac{1}{2}\begin{bmatrix} 2 & -4 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}$

6. $f(x) = 2x^2 - 3x$

Differentiating w.r.t. *x*,

$f'(x) = 4x - 3$

(a) For strictly increasing

$f'(x) > 0$

$\Rightarrow \quad 4x - 3 > 0 \quad \Rightarrow \quad x > \frac{3}{4}$ or $x \in \left(\frac{3}{4}, \infty\right)$

(b) For strictly decreasing

$f'(x) < 0$

$\Rightarrow \quad 4x - 3 < 0 \quad \Rightarrow \quad x < \frac{3}{4}$ or $x \in \left(-\infty, \frac{3}{4}\right)$

7. Differentiating $x^2 = 4y$ with respect to *x*, we get :

$$\frac{dy}{dx} = \frac{x}{2}$$

Let (*h*, *k*) be the coordinates of the foot of the normal to the curve $x^2 = 4y$. Now, slope of the tangent at (h, k) is given by $\left.\frac{dy}{dx}\right]_{(h,k)} = \frac{h}{2}$

Hence, slope of the normal at (*h*, *k*) = $\frac{-2}{h}$

Therefore, the equation of normal at (*h*, *k*) is $y - k = \frac{-2}{h}(x - h)$ …(1)

Since it passes through the point (1, 2), we have

$2 - k = \frac{-2}{h}(1 - h)$ or $k = 2 + \frac{2}{h}(1 - h)$ …(2)

Since (*h*, *k*) lies on the curve $x^2 = 4y$, we have

$h^2 = 4k$ …(3)

Form (2) and (3), we have h = 2 and k = 1. Substituting the values of *h* and *k* in (1), we get the required equation of normal as $y - 1 = \frac{-2}{2}(x - 2)$ or $x + y = 3$

8. If the profit function is $p(x) = 41 - 72x - 18x^2$

So differentiating w.r.t. *x*

we get : $p'(x) = -72 - 36x$

For maxima or minima, putting $p'(x) = 0 \quad \Rightarrow \quad -72 - 36x = 0 \quad \Rightarrow \quad x = -2$

Differentiating again $p''(x) = -36$

i.e. $p''(x) < 0$

So *p*(*x*) is maximum at *x* = –2

and maximum profit is $p(-2) = 41 - 72(-2) - 18(-2)^2 = 113$

So the company can make a maximum profit of Rs. 113.

9. Comparing the given equation with the standard form $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$

We observe that $x_1 = -3, y_1 = 5, z_1 = -6; a = 2, b = 4, c = 2$

Thus, the required line passes through the point (–3, 5, –6) and is parallel to the vector $2\hat{i} + 4\hat{j} + 2\hat{k}$. Let $\vec{r}$ be the position vector of any point on the line, then the vector equation of the line is given by $\vec{r} = \left(-3\hat{i} + 5\hat{j} - 6\hat{k}\right) + \lambda\left(2\hat{i} + 4\hat{j} + 2\hat{k}\right)$.

10. **(i)** $\because$ $A \subseteq B$

$\Rightarrow$ $A \cap B = A$ and $P(A \cap B) = P(A)$

So $P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)} = \frac{P(A)}{P(A)} = 1$

(ii) $\because$ $A \cap B = \phi$

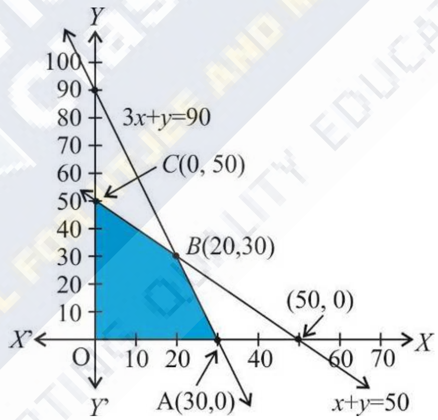
$\Rightarrow$ $P(A \cap B) = 0$

So $P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)} = \frac{0}{P(A)} = 0 \; (\because P(A) \neq 0)$

11. The shaded region in fig. is the feasible region determined by the system of constraints (2) to (4). We observe that the feasible region *OABC* is bounded. So, we now use corner point method to determine the maximum value of *Z*.

The coordinates of the corner points *O*, *A*, *B* and *C*, are (0, 0), (30, 0), (20, 30) and (0, 50) respectively. Now we evaluate *Z* at each corner point.

Corner Point	Corresponding value of Z	
(0, 0)	0	
(30, 0)	120 $\leftarrow$	Maximum
(20, 30)	110	
(0, 50)	50	



Hence, maximum value of Z is 120 at the point (30, 0).

12. We have $x^2 - 6x + 13 = x^2 - 6x + 3^2 - 3^2 + 13 = (x-3)^2 + 4$

So, $\int \frac{dx}{x^2 - 6x + 13} = \int \frac{1}{(x-3)^2 + 2^2} dx$

Let $x - 3 = t$. then $dx = dt$

Therefore, $\int \frac{dx}{x^2 - 6x + 13} = \int \frac{dt}{t^2 + 2^2} = \frac{1}{2}\tan^{-1}\frac{t}{2} + C = \frac{1}{2}\tan^{-1}\frac{x-3}{2} + C$

[SECTION C]

13. Given that $0 < x < 1$

and $\sqrt{1+x^2}\left[\left\{x\cos\left(\cot^{-1}x\right)+\sin\left(\cot^{-1}x\right)\right\}^2-1\right]^{1/2}=2x$

$\Rightarrow \quad \sqrt{1+x^2}\left[\left\{x\cos\left(\cos^{-1}\frac{x}{\sqrt{1+x^2}}\right)+\sin\left(\sin^{-1}\frac{1}{\sqrt{1+x^2}}\right)\right\}^2-1\right]^{1/2}=2x$

$\Rightarrow \quad \sqrt{1+x^2}\left[\left\{\frac{x^2}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+x^2}}\right\}^2-1\right]^{1/2}=2x \Rightarrow \quad \sqrt{1+x^2}\left[\left\{\frac{x^2+1}{\sqrt{1+x^2}}\right\}^2-1\right]^{1/2}=2x$

$\Rightarrow \quad \sqrt{1+x^2}\left[1+x^2-1\right]^{1/2}=2x \qquad \Rightarrow \quad \sqrt{1+x^2}.|x|=2x$

$\Rightarrow \quad \sqrt{1+x^2}.x=2x\left(\because\ 0<x<1\right) \qquad \Rightarrow \quad \sqrt{1+x^2}=2$

$\Rightarrow \quad 1+x^2=4 \qquad \Rightarrow \quad x=\pm\sqrt{3}$ So solution is $x=\sqrt{3}\left(\because 0<x<1\right)$

14. Given that $A=\begin{bmatrix}0 & -\tan\frac{\alpha}{2}\\ \tan\frac{\alpha}{2} & 0\end{bmatrix}$ and $I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

So, L.H.S = I + A $=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\begin{bmatrix}0 & -\tan\frac{\alpha}{2}\\ \tan\frac{\alpha}{2} & 0\end{bmatrix}=\begin{bmatrix}1 & -\tan\frac{\alpha}{2}\\ \tan\frac{\alpha}{2} & 1\end{bmatrix}$

and R.H.S $=\left(I-A\right)\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}$

$=\begin{bmatrix}1 & \tan\frac{\alpha}{2}\\ -\tan\frac{\alpha}{2} & 1\end{bmatrix}\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}=\begin{bmatrix}\cos\alpha+\sin\alpha.\tan\frac{\alpha}{2} & -\sin\alpha+\cos\alpha.\tan\frac{\alpha}{2}\\ -\cos\alpha.\tan\frac{\alpha}{2}+\sin\alpha & \sin\alpha.\tan\frac{\alpha}{2}+\cos\alpha\end{bmatrix}$

Now putting $\sin\alpha=\dfrac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$ and $\cos\alpha=\dfrac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}=\begin{bmatrix}1 & -\tan\frac{\alpha}{2}\\ \tan\frac{\alpha}{2} & 1\end{bmatrix}$

L.H.S = R.H.S
Hence proved.

OR

Applying operations $R_2\to R_2-2R_1$ and $R_3\to R_3-3R_1$ to the given determinant Δ, we have

$$\Delta=\begin{vmatrix}a & a+b & a+b+c\\ 0 & a & 2a+b\\ 0 & 3a & 7a+3b\end{vmatrix}$$

Now applying $R_3\to R_3-3R_2$, we get : $\Delta=\begin{vmatrix}a & a+b & a+b+c\\ 0 & a & 2a+b\\ 0 & 0 & a\end{vmatrix}$

Expanding along C_1, we obtain $\Delta=a\begin{vmatrix}a & 2a+b\\ 0 & a\end{vmatrix}+0+0=a\left(a^2-0\right)=a\left(a^2\right)=a^3$

15. $y=\left(\tan^{-1}x\right)^2$ …(1)

Differentiating w.r.t. x

$$\frac{dy}{dx}=2\left(\tan^{-1}x\right).\frac{1}{1+x^2}$$

$$\left(1+x^2\right)^2\left(\frac{dy}{dx}\right)^2=4\left(\tan^{-1}x\right)^2$$

$$\left(1+x^2\right)^2\left(\frac{dy}{dx}\right)^2=4y \qquad \text{(from equation (1))}$$

Differentiating again w.r.t. to x

$$2\left(1+x^2\right)\times 2x\left(\frac{dy}{dx}\right)^2+\left(1+x^2\right)^2\times 2\frac{dy}{dx}\times\frac{d^2y}{dx^2}=4\frac{dy}{dx}$$

$$2\frac{dy}{dx}\left[2\left(1+x^2\right)\left(\frac{dy}{dx}\right)x+\left(1+x^2\right)^2\frac{d^2y}{dx^2}\right]=4\frac{dy}{dx}$$

$$\left(1+x^2\right)^2\frac{d^2y}{dx^2}+2x\left(1+x^2\right)\frac{dy}{dx}=2 \qquad \left[\frac{dy}{dx}\neq 0\right]$$

OR

Differentiate the given function f w.r.t. x.

$$f\left(x\right)=\left(\log x\right)^x+x^{\log x}$$

OR

$$f\left(x\right)=\left(\ln x\right)^x+\left(x\right)^{\ln x}$$

Let $u=\left(\ln x\right)^x$ and $v=\left(x\right)^{\ln x}$

Taking log on both sides

$\ln u=x.\ln\left(\ln x\right)$

Differentiating w.r.t. x

$$\frac{1}{u}\times\frac{du}{dx}=x.\frac{1}{\ln x}.\frac{1}{x}+\ln\left(\ln x\right)$$

$$\frac{du}{dx}=u\left[\frac{1}{\ln x}+\ln\left(\ln x\right)\right] \qquad \text{…..(1)}$$

and $v=\left(x\right)^{\ln x}$

Taking log on both sides

$$\ln v=\ln x.\ln x=\left(\ln x\right)^2$$

Differentiating w.r.t x

$$\frac{1}{v}\frac{dv}{dx}=2\ln x\times\frac{1}{x} \quad\Rightarrow\quad \frac{dv}{dx}=\left(x\right)^{\ln x}\times\frac{2\ln x}{x}$$

$$\frac{dv}{dx}=2\left(\ln x\right).\left(x\right)^{\ln x-1} \qquad \text{…(2)}$$

So $f\left(x\right)=u+v$

Differentiating w.r.t. x

$$f'\left(x\right)=\frac{du}{dx}+\frac{dv}{dx}$$

$$\Rightarrow \quad f'\left(x\right)=\left(\ln x\right)^x\left[\frac{1}{\ln x}+\ln\left(\ln x\right)\right]+2\left(\ln x\right)\left(x\right)^{\ln x-1} \qquad \text{(From (1) and (2))}$$

16. Let $y = \sin\phi$

Then $dy = \cos\phi \, d\phi$

Therefore, $\int \frac{(3\sin\phi - 2)\cos\phi}{5 - \cos^2\phi - 4\sin\phi} d\phi = \int \frac{(3y-2)\,dy}{5-(1-y^2)-4y}$

$$= \int \frac{3y-2}{y^2-4y+4} dy = \int \frac{3y-2}{(y-2)^2} = I\,(say)$$

Now we write $\frac{3y-2}{(y-2)^2} = \frac{A}{y-2} + \frac{B}{(y-2)^2}$ [by Table 7.2 (2)]

Therefore $3y - 2 = A(y-2) + B$

Comparing the coefficients of y and constant term, we get A = 3 and B –2A = –2, which gives A = 3 and B = 4
Therefore, the required integral given by

$$I = \int \left[\frac{3}{y-2} + \frac{4}{(y-2)^2} \right] dy = 3\int \frac{dy}{y-2} + 4\int \frac{dy}{(y-2)^2} = 3\log_e |y-2| + 4\left(-\frac{1}{(y-2)} \right) + C$$

$$= 3\log_e (2 - \sin\phi) + \frac{4}{2-\sin\phi} + C$$ (since, $2 - \sin\phi$ is always positive)

17. Let $I = \int_0^{\frac{\pi}{2}} \log\sin x dx$

$$I = \int_0^{\frac{\pi}{2}} \log\sin\left(\frac{\pi}{2} - x\right) dx = \int_0^{\frac{\pi}{2}} \log\cos x dx$$

Adding the two values of I, we get

$$2I = \int_0^{\frac{\pi}{2}} (\log\sin x + \log\cos x)\,dx$$

$$= \int_0^{\frac{\pi}{2}} (\log\sin x\cos x + \log 2 - \log 2)\,dx$$ (by adding and subtracting log2)

$$= \int_0^{\frac{\pi}{2}} \log\sin 2x dx - \int_0^{\frac{\pi}{2}} \log 2 dx$$

Put $2x = t$ in the first integral. Then $2dx = dt$, when $x = 0, t = 0$ and when $x = \frac{\pi}{2}$, $t = \pi$

Therefore $2I = \frac{1}{2}\int_0^{\pi} \log\sin t dt - \frac{\pi}{2}\log 2$

$$= \frac{2}{2}\int_0^{\frac{\pi}{2}} \log\sin t dt - \frac{\pi}{2}\log 2$$ [As $\sin(\pi - t) = \sin t$]

$$= \int_0^{\frac{\pi}{2}} \log\sin x dx - \frac{\pi}{2}\log 2$$ (by changing variable t to x)

$$= I - \frac{\pi}{2}\log 2$$

Hence $\int_0^{\frac{\pi}{2}} \log\sin x dx = \frac{-\pi}{2}\log 2.$

OR

Evaluate the definite integrals in $\int_1^4 \left[|x-1| + |x-2| + |x-3| \right] dx$

OR

$$\int_1^4 \left(|x-1|+|x-2|+|x-3|\right)dx$$

$$= \int_1^2 \left((x-1)-(x-2)-(x-3)\right)dx + \int_2^3 \left((x-1)+(x-2)-(x-3)\right)dx + \int_3^4 \left((x-1)+(x-2)+(x-3)\right)dx$$

$$= \int_1^2 (x-1-x+2-x+3)\,dx + \int_2^3 (x-1+x-2-x+3)\,dx + \int_3^4 (x-1+x-2+x-3)dx$$

$$= \int_1^2 (-x+4)\,dx + \int_2^3 (x)\,dx + \int_3^4 (3x-6)\,dx = \left[-\frac{x^2}{2}+4x\right]_1^2 , + \left[\frac{x^2}{2}\right]_2^3 + \left[\frac{3x^2}{2}-6x\right]_3^4 = \frac{19}{2}$$

18. **(a)** Note that if θ is the angle between AB and CD, then θ is also the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$.

Now $\overrightarrow{AB}$ = Position vector of B – position vector of A $= (2\hat{i}+5\hat{j})-(\hat{i}+\hat{j}+\hat{k}) = \hat{i}+4\hat{j}-\hat{k}$

Therefore $\left|\overrightarrow{AB}\right| = \sqrt{(1)^2+(4)^2+(-1)^2} = 3\sqrt{2}$

Similarly $\overrightarrow{CD} = -2\hat{i}-8\hat{j}+2\hat{k}$ and $\left|\overrightarrow{CD}\right| = 6\sqrt{2}$

Thus $\cos\theta = \dfrac{\overrightarrow{AB}.\overrightarrow{CD}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{CD}\right|} = \dfrac{1(-2)+4(-8)+(-1)(2)}{(3\sqrt{2})(6\sqrt{2})} = \dfrac{-36}{36} = -1$

Since $0 \le \theta \le \pi$, it follows that $\theta = \pi$. This shows that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear

Alternatively, $\overrightarrow{AB} = -\dfrac{1}{2}\overrightarrow{CD}$ which implies that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear vectors.

(b) Since $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, we have $\vec{a}.\left(\vec{a}+\vec{b}+\vec{c}\right)=0$

Or $\vec{a}.\vec{a}+\vec{a}.\vec{b}+\vec{a}.\vec{c}=\vec{0}$

Therefore $\vec{a}.\vec{b}+\vec{a}.\vec{c} = -|\vec{a}|^2 = -1$ …(1) Again $\vec{b}.\left(\vec{a}+\vec{b}+\vec{c}\right)=0$

Or $\vec{a}.\vec{b}+\vec{b}.\vec{c} = -\left|\vec{b}\right|^2 = -16$ …(2)

Similarly $\vec{a}.\vec{c}+\vec{b}.\vec{c} = -4$ …(3)

Adding (1), (2) and (3), we have $2\left(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{a}.\vec{c}\right) = -21$

Or $2\mu = -21, i.e., \mu = \dfrac{-21}{2}$

19. The given function is $y = e^{ax}\left[c_1\cos bx + c_2\sin bx\right]$ …(1)

Differentiating both sides of equation (1) with respect to x, we get :

$$\frac{dy}{dx} = e^{ax}\left[-bc_1\sin bx + bc_2\cos bx\right] + \left[c_1\cos bx + c_2\sin bx\right]e^{ax}.a$$

Or $\dfrac{dy}{dx} = e^{ax}\left[(bc_2+ac_1)\cos bx + (ac_2-bc_1)\sin bx\right]$ …(2)

Differentiating both sides of equation (2) with respect to x, we get :

$$\frac{d^2y}{dx^2} = e^{\alpha x}\left[(bc_2+ac_1)(-b\sin bx)+(ac_2-bc_1)(b\cos bx)\right] + \left[(bc_2+ac_1)\cos bx+(ac_2-bc_1)\sin bx\right]e^{ax}.a$$

$$= e^{ax}\left[\left(a^2c_2-2abc_1-b^2c_2\right)\sin bx + \left(a^2c_1+2abc_2-b^2c_1\right)\cos bx\right]$$

Substituting the values of $\frac{d^2y}{dx^2}, \frac{dy}{dx}$ and y in the given differential equation, we get :

L.H.S. $= e^{ax}\left[a^2c_2 - 2abc_1 - b^2c_2)\sin bx + \left(a^2c_1 + 2abc_2 - b^2c_1\right)\cos bx\right]$

$-2ae^{ax}\left[\left(bc_2 + ac_1\right)\cos bx + \left(ac_2 - bc_1\right)\sin bx\right] + \left(a^2 + b^2\right)e^{ax}\left(c_1\cos bx + c_2\sin bx\right)$

$$= e^{ax}\begin{bmatrix}\left(a^2c_2 - 2abc_1 - b^2c_2 - 2a^2c_2 + 2abc_1 + a^2c_2 + b^2c_2\right)\sin bx \\ +\left(a^2c_1 + 2abc_2 - b^2c_1 - 2abc_2 - 2a^2c_1 + a^2c_1 + b^2c_1\right)\cos bx\end{bmatrix}$$

$= e^{ax}\left[0\times\sin bx + 0\cos bx\right] = e^{ax}\times 0 = 0 = R.H.S.$

20. Given that $|\vec{a}| = |\vec{b}| = |\vec{c}| = k\,(say)$

and $\quad \vec{a}.\vec{b} = \vec{b}.\vec{c} = \vec{c}.\vec{a} = 0$

Let $\quad \vec{d} = \vec{a} + \vec{b} + \vec{c}$

Now $\quad |\vec{d}|^2 = |\vec{a} + \vec{b} + \vec{c}|^2$

$= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2\left(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}\right) = k^2 + k^2 + k^2$

$|\vec{d}|^2 = 3k^2 \Rightarrow |\vec{d}| = \sqrt{3}k$

Let θ_1 be the angle between $\vec{a}$ and $\vec{d}$ so $\vec{a}.\vec{d} = |\vec{a}||\vec{d}|\cos\theta_1$

$\vec{a}.\left(\vec{a} + \vec{b} + \vec{c}\right) = k.\sqrt{3}k\cos\theta_1$

$\Rightarrow \quad |\vec{a}|^2 + \vec{a}.\vec{b} + \vec{a}.\vec{c} = \sqrt{3}k^2\cos\theta_1 \quad \Rightarrow \quad \cos\theta_1 = \frac{k^2}{\sqrt{3}k^2} = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad \theta_1 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

Similarly the angle between $\vec{b},\vec{d}$ and $\vec{c},\vec{d}$ is $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

So we can say that $\vec{d}$ is equally inclined to the vectors $\vec{a},\vec{b}$ and $\vec{c}$, and the angle is $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

21. The Probability distribution of X is

X	0	1	2	3	4
P(X)	0.1	k	2k	2k	k

(a) We know that $\sum_{i=1}^{n} p_i = 1$

Therefore $0.1 + k + 2k + 2k + k = 1$

i.e. $k = 0.15$

(b) P(you study at least two hours) $= P\left(X \geq 2\right)$

$= P\left(X = 2\right) + P\left(X = 3\right) + P\left(X = 4\right)$

$= 2k + 2k + k = 5k = 5\times 0.15 = 0.75$

P(you study exactly two hours) $= P\left(X = 2\right)$

$= 2k = 2\times 0.15 = 0.3$

P (you study at most two hours) $= P\left(X \leq 2\right)$

$= P\left(X = 0\right) + P\left(X = 1\right) + P\left(X = 2\right)$

$= 0.1 + k + 2k = 0.1 + 3k = 0.1 + 3\times 0.15$

$= 0.55$

22. Let E be the event that the doctor visits the patient late and let T_1, T_2, T_3, T_4 be the events that the doctor comes by train, bus, scooter and other means of transport respectively.

Then $P(T_1)=\frac{3}{10}, P(T_2)=\frac{1}{5}, P(T_3)=\frac{1}{10}$ and $P(T_4)=\frac{2}{5}$ (given)

$P(E/T_1)=$ Probability that the doctor arriving late comes by train $=\frac{1}{4}$

Similarly, $P(E/T_2)=\frac{1}{3}, P(E|T_3)=\frac{1}{12}$ and $P(E|T_4)=0$, since he is not late if he comes by other means of transport.

Therefore, by Bayes' Theorem, we have

$P(T_1|E)=$ Probability that the doctor arriving late comes by train

$$=\frac{P(T_1)P(\text{E}|\text{T}_1)}{P(T_1)P(E|T_1)+P(T_2)P(E|T_2)+P(T_3)P(E|T_3)+P(T_4)P(E|T_4)}$$

$$=\frac{\frac{3}{10}\times\frac{1}{4}}{\frac{3}{10}\times\frac{1}{4}+\frac{1}{5}\times\frac{1}{3}+\frac{1}{10}\times\frac{1}{12}+\frac{2}{5}\times 0}=\frac{3}{40}\times\frac{120}{18}=\frac{1}{2}$$

Hence, the required probability is $\frac{1}{2}$.

23. Let x and y be the number of items M and N respectively.

Total profit on the production = Rs (600 x + 400 y)

Mathematical formulation of the given problem is as follows.

Maximise $Z = 600x + 400y$

Subject to the constraints

$x+2y\le 12$ (constraint on Machine I) ...(1)

$2x+y\le 12$ (constraint on Machine II) …(2)

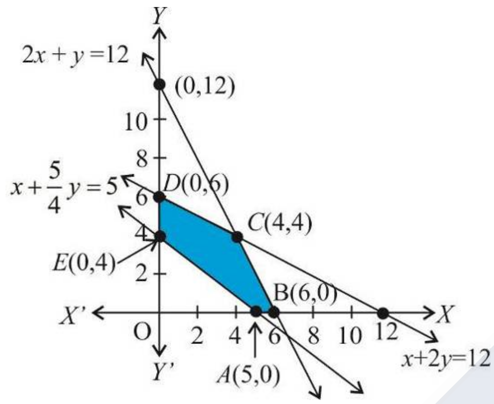
$x+\frac{5}{4}y\ge 5$ (constraint on Machine III) …(3)

$x\ge 0, y\ge 0$ …(4)

Let us draw the graph of constraint (1) to (4). *ABCDE* is the feasible region (shaded) as shown in figure. determined by the constraint (1) to (4). Observe that the feasible region is bounded, coordinates of the corner points A, B, C, D and E are (5, 0) (6, 0), (4, 4), (0, 6) and (0, 4) respectively.

Let us evaluate $Z = 600x + 400y$ at these corner points.

Corner Point	Z = 600x + 400y
(5, 0)	3000
(6, 0)	3600
(4, 4)	4000 $\leftarrow$ maximum
(0, 6)	2400
(0, 4)	1600



We see that the point (4, 4) is giving the maximum value of Z. Hence, the manufacturer has to produce 4 units of each item to get the maximum profit of Rs 4000.

[SECTION D]

24. Let first, second and third numbers be denoted by *x*, *y* and *z*, respectively. Then, according to given conditions, We have

$x + y + z = 6$

$y + 3z = 11$

$x + z = 2y$ or $x - 2y + z = 0$

This system can be written as $AX = B$, where $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -2 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$

Here $|A| = 1(1+6) - (0-3) + (0-1) = 9 \neq 0.$ Now we find adj A

$a_{11} = 1(1+6) = 7,\quad a_{12} = -(0-3) = 3,\quad a_{13} = -1$

$a_{21} = -1(1+2) = -3,\quad a_{22} = 0,\quad a_{23} = -(-2-1) = 3$

$a_{31} = (3-1) = 2,\quad a_{32} = -(3-0) = -3,\quad a_{33} = (1-0) = 1$

Hence $adj\,A = \begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}$ Thus $A^{-1} = \frac{1}{|A|} adj(A) = \frac{1}{9}\begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}$

Since $X = A^{-1}B$

$X = \frac{1}{9}\begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$ Or $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 42-33+0 \\ 18+0+0 \\ -6+33+0 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 \\ 18 \\ 27 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Thus $x = 1, y = 2, z = 3$

25. It is given that * : $R \times R \to R$ is defined as $a * b = |a - b| \forall a, b \in R$

So $a * b = |a - b| = |b - a| = b * a \forall a, b \in R$

So $a * b = b * a \forall a, b \in R$

So * is commutative operation

Now to check associativity

$a*(b*c) = a*|b-c| = |a - |b-c|| \forall a,b,c \in R$ and $(a*b)*c = |a-b|*c = ||a-b| - c| \forall a,b,c \in R.$

Clearly $a*(b*c) \neq (a*b)*c$

So * is not associative

Again $o: R \times R \to R$ is defined as a $aob = a \forall a, b \in R$

$aob = a$ and $boa = b \forall a, b \in R$

So operation ‘o’ is not commutative.

and $ao(boc) = aob = a \forall a, b, c \in R$

and $(aob)oc = aoc = a \forall \ a, b, c \in R$

So operation ‘o’ is associative.

Further $a*(boc) = a*b [\because boc = b] = |a-b|$ and $(a*b)o(a*c) = |a-b|o|a-c| = |a-b|$

So $a*(boc) = (a*b)0(a*c) \forall a, b, c \in R$

Now to check ‘O’ is distributive over ‘*’

$ao(b*c) = ao|b-c| = a$

and $(aob)*(a0c) = a*a = |a-a| = 0 \forall a, b, c \in R$

here $ao(b*c) \neq (aob)*(aoc)$

So ‘o’ is not distributive over *R*.

OR

$f: R^+ \to (-5, \infty)$

$f(x) = 9x^2 + 6x - 5 = y$

$\Rightarrow \quad 9x^2 + 6x - 5 - y = 0 \quad \Rightarrow \quad x = \dfrac{-6 \pm \sqrt{36 - 4 \times 9 \times (-5-y)}}{2 \times 9}$

$\Rightarrow \quad x = \dfrac{-6 \pm 6\sqrt{1+5+y}}{18} = \dfrac{-1 \pm \sqrt{y+6}}{3}$

$\because \quad x \in R+$ so neglecting minus sign

$\Rightarrow \quad x = \dfrac{-1+\sqrt{y+6}}{3} = f^{-1}(y)$

Hence $f^{-1}(y) = \dfrac{\sqrt{y+6}-1}{3}$

(i) $\quad f^{-1}(3) = \dfrac{\sqrt{3+6}-1}{3} = \dfrac{3-1}{3} = \dfrac{2}{3}$

(ii) $\quad f^{-1}(y) = \dfrac{4}{3}$

$\Rightarrow \quad \dfrac{\sqrt{y+6}-1}{3} = \dfrac{4}{3}$

$\Rightarrow \quad \sqrt{y+6} = 5$

$\Rightarrow \quad y = 19$

26. It is given that height of cone is h and semi vertical angle is α. Let r be the radius and h' be the height of cylinder.

Now in ΔAEH, $\tan\alpha = \dfrac{EH}{AH} = \dfrac{r}{AH}$

$\Rightarrow \quad AH = r\cot\alpha$, so $AI = AH + HI$

$\Rightarrow \quad h = r\cot\alpha + h'$

$\Rightarrow \quad h' = h - r\cot\alpha \quad$ …..(1)

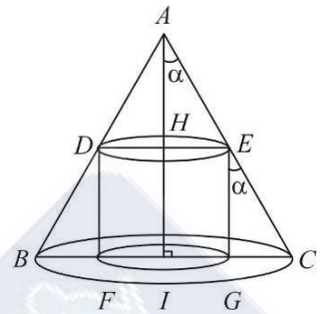
Now let V be the volume of cylinder

So $V = \pi r^2 h' = \pi r^2 (h - r\cot\alpha)$ (from equation (1))

$V = \pi h r^2 - \pi r^3 \cot\alpha$

Differentiating w.r.to r

$$\frac{dV}{dr} = 2\pi h r - 3\pi r^2 \cot\alpha \quad \ldots(2)$$



For maxima and minima, putting $\frac{dV}{dr} = 0$

$2\pi h r - 3\pi r^2 \cot\alpha = 0$

$\Rightarrow \quad 2h = 3r\cot\alpha \quad \Rightarrow \quad r = \frac{2h}{3}\tan\alpha$

and $\quad h' = h - \frac{2h}{3}\tan\alpha.\cot\alpha$

$\Rightarrow \quad h' = \frac{1}{3}h$ which is we have to prove.

Again differentiating equation (2) w.r.t. r

$$\frac{d^2V}{dr^2} = 2\pi h - 6\pi r\cot\alpha = 2\pi h - 6\pi.\frac{2h}{3}\tan\alpha.\cot\alpha = -2\pi h$$

$\Rightarrow \quad \frac{d^2V}{dr^2} < 0$

So volume of cylinder is maxima.

and maximum volume = $\pi r^2 h' = \pi \times \frac{4}{9}h^2\tan^2\alpha \times \frac{1}{3}h = \frac{4}{27}\pi h^3 \tan^2\alpha.$ (Hence proved)

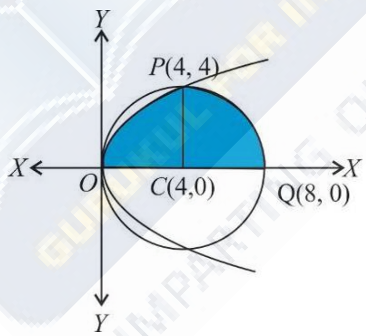
27. The given equation of the circle $x^2 + y^2 = 8x$ can be expressed as $(x-4)^2 + y^2 = 16$. Thus, the center of the circle is (4, 0) and radius is 4. Its intersection with the parabola $y^2 = 4x$ gives

$x^2 + 4x = 8x$

or $\quad x^2 - 4x = 0 \quad$ or $\quad x(x-4) = 0 \quad$ or $\quad x = 0, x = 4$

Thus the points of intersection of these two curves are $O(0, 0)$ and P(4, 4) above the x-axis.

From the fig the required area of the region *OPQCO* included between these two curves above x-axis is



= (area of the region *OCPO*) + (area of the region *PCQP*)

$$= \int_0^4 y\,dx + \int_4^8 y\,dx = 2\int_0^4 \sqrt{x}\,dx + \int_4^8 \sqrt{4^2 - (x-4)^2}\,dx = 2\times\frac{2}{3}\left[x^{3/2}\right]_0^4 + \int_0^4 \sqrt{4^2 - t^2}\,dt \text{ where } x - 4 = t$$

$$= \frac{32}{3} + \left[\frac{t}{2}\sqrt{4^2 - t^2} + \frac{1}{2}\times 4^2 \times \sin^{-1}\frac{t}{4}\right]_0^4$$

$$= \frac{32}{3} + \left[\frac{4}{2}\times 0 + \frac{1}{2}\times 4^2 \times \sin^{-1} 1\right] = \frac{32}{3} + \left[0 + 8\times\frac{\pi}{2}\right] = \frac{32}{3} + 4\pi = \frac{4}{3}(8 + 3\pi)$$

OR

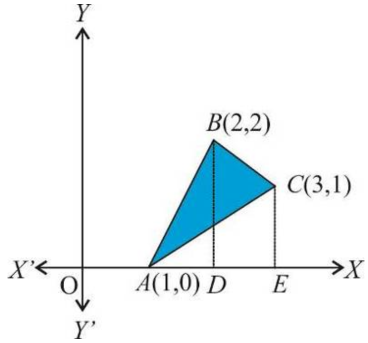
Using integration find the area of region bounded by the triangle whose vertices are (1, 0), (2, 2) and (3, 1).

OR

Let A (1, 0), B (2, 2) and C (3, 1) be the vertices of a triangle *ABC*.

Area of ΔABC = Area of ΔABD + area of trapezium BDEC – Area of ΔAEC

Now equation of the sides *AB*, *BC* and *CA* are given by



$y = 2(x-1), y = 4 - x, y = \frac{1}{2}(x-1),$ respectively

Hence, area of $\Delta ABC = \int_1^2 2(x-1)dx + \int_2^3 (4-x)\,dx - \int_1^3 \frac{x-1}{2}dx$

$$= 2\left[\frac{x^2}{2} - x\right]_1^2 + \left[4x - \frac{x^2}{2}\right]_2^3 - \frac{1}{2}\left[\frac{x^2}{2} - x\right]_1^3$$

$$= 2\left[\left(\frac{2^2}{2} - 2\right) - \left(\frac{1}{2} - 1\right)\right] + \left[\left(4\times 3 - \frac{3^2}{2}\right) - \left(4\times 2 - \frac{2^2}{2}\right)\right] - \frac{1}{2}\left[\left(\frac{3^2}{2} - 3\right) - \left(\frac{1}{2} - 1\right)\right] = \frac{3}{2}$$

28. The given differential equation can be written as

$$\frac{dx}{dy} = \frac{2xe^{\frac{x}{y}} - y}{2ye^{\frac{x}{y}}} \qquad \ldots(1)$$

Let $F(x, y) = \frac{2xe^{\frac{x}{y}} - y}{2ye^{\frac{x}{y}}}$

Then $F(\lambda x, \lambda y) = \frac{\lambda\left(2xe^{\frac{x}{y}} - y\right)}{\lambda\left(2ye^{\frac{x}{y}}\right)} = \lambda^0\left[F(x, y)\right]$

Thus, F(x, y) is a homogeneous function of degree zero. Therefore, the given differential equation is a homogeneous differential equation.

To solve it, we make the substitution

$$x = vy \qquad \ldots(2)$$

Differentiating equation (2) with respect to *y*, we get :

$$\frac{dx}{dy} = v + y\frac{dv}{dy}$$

Substituting the value of x and $\frac{dx}{dy}$ in equation (1), we get :

$$v + y\frac{dv}{dy} = \frac{2ve^v - 1}{2e^v} \quad \text{or} \quad y\frac{dv}{dy} = \frac{2ve^v - 1}{2e^v} - v \quad \text{or} \quad y\frac{dv}{dy} = \frac{-1}{2e^v}$$

$$\text{or} \quad 2e^v dv = \frac{-dy}{y} \quad \text{or} \quad \int 2e^v.dv = -\int \frac{dy}{y} \quad \text{or} \quad 2e^v = -\ln|y| + C$$

and replacing v by $\frac{x}{y}$, we get :

$$2e^{\frac{x}{y}} + \ln|y| = C \qquad \ldots(3)$$

Substituting $x = 0$ and $y = 1$ in equation (3), we get :

$$2e^0 + \ln|1| = C \Rightarrow C = 2$$

Substituting the value of C in equation (3), we get :

$$2e^{\frac{x}{y}} + \ln|y| = 2$$

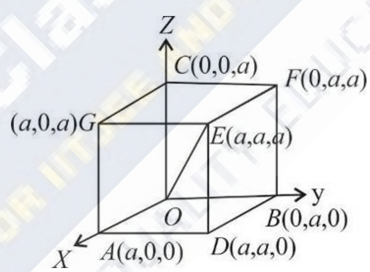
Which is the particular solution of the given differential equation.

29. A cube is a rectangular parallelepiped having equal length, breadth and height. Let $OADBFEGC$ be the cube with each side of length a unit. The four diagonals are *OE, AF, BG* and *CD.*

The direction cosines of the diagonal OE which is the line joining two points O and E are

$$\frac{a-0}{\sqrt{a^2+a^2+a^2}}, \frac{a-0}{\sqrt{a^2+a^2+a^2}}, \frac{a-0}{\sqrt{a^2+a^2+a^2}}$$

i.e. $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$



Similarly, the direction cosines of AF, BG and CD are $\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}; \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$, respectively

Let l, m, n be the direction cosines of the given line which makes angles $\alpha, \beta, \gamma, \delta$ with OE, AF, BG, CD, respectively. Then :

$$\cos\alpha = \frac{1}{\sqrt{3}}(l+m+n); \cos\beta = \frac{1}{\sqrt{3}}(-l+m+n); \ \cos\gamma = \frac{1}{\sqrt{3}}(l-m+n); \cos\delta = \frac{1}{\sqrt{3}}(l+m-n)$$

Squaring and adding, we get :

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$$

$$= \frac{1}{3}\left[(l+m+n)^2 + (-l+m+n)^2 + (l-m+n)^2 + (l+m-n)^2\right]$$

$$= \frac{1}{3}\left[4\left(l^2+m^2+n^2\right)\right] = \frac{4}{3} \text{ (as } l^2+m^2+n^2 = 1)$$

OR

Find the equation of the plane which contains the line of intersection of the planes $\vec{r}.\left(\hat{i}+2\hat{j}+3\hat{k}\right)-4=0, \vec{r}.\left(2\hat{i}+\hat{j}-\hat{k}\right)+5=0$ and which is perpendicular to the plane $\vec{r}.\left(5\hat{i}+3\hat{j}-6\hat{k}\right)+8=0$ .Find whether the plane thus obtained contains the line $6x-12=-6y-6=25z-10$.

OR

We know that equation of any plane which pass through the line of intersection of planes $\vec{r}.\vec{n}_1=d_1$, and $\vec{r}.\vec{n}_2=d_2$ is $\vec{r}.\left(\vec{n}_1+\lambda\vec{n}_2\right)=d_1+\lambda d_2$ here $\vec{n}_1=\hat{i}+2\hat{j}+3\hat{k}$, $\vec{n}_2=2\hat{i}+\hat{j}-\hat{k}, d_1=4, d_2=-5$

So equation of required plane is

$$\vec{r}.\left((1+2\lambda)\hat{i}+(2+\lambda)\hat{j}+(3-\lambda)\hat{k}\right)=4-5\lambda \qquad \ldots.(1)$$

But it is given that above plane is perpendicular to plane $\vec{r}.\left(5\hat{i}+3\hat{j}-6\hat{k}\right)+8=0$

So $\quad 5(1+2\lambda)+3(2+\lambda)-6(3-\lambda)=0 \quad \Rightarrow \quad \lambda=7/19$

Putting the value of λ in equation (1), we get required plane is $\vec{r}.\left[\left(1+\frac{14}{19}\right)\hat{i}+\left(2+\frac{7}{19}\right)\hat{j}+\left(3-\frac{7}{19}\right)\hat{k}\right]=4-\frac{35}{19}$

$$\Rightarrow \quad \vec{r}.\left(33\hat{i}+45\hat{j}+50\hat{k}\right)=41 \quad \text{or} \quad 33x+45y+50z=41 \qquad \ldots(2)$$

Now we check the given line $6x-12=-6y-6=25z-10$ lie on plane (2) or not.

So the line $6(x-2)=-6(y+1)=25\left(z-\frac{2}{5}\right)$

$$\Rightarrow \frac{x-2}{1/6}=\frac{y+1}{-1/6}=\frac{z-2/5}{1/25}$$

Passes through $\left(2,-1,\frac{2}{5}\right)$ whose d.r's are $\left(\frac{1}{6},\frac{-1}{6},\frac{1}{25}\right)$

Here $\quad 33\times\frac{1}{6}-45\times\frac{1}{6}+50\times\frac{1}{25}=\frac{11}{2}-\frac{15}{2}+\frac{4}{2}=0$ and $\quad 33\times2-45\times1+50\times\frac{2}{5}=66-45+20=41\neq0$

So the given line not lie on the plane 2.

Vidyamandir Classes
Gurukul for IITJEE & Medical Preparation

Answers to Sample Paper

Class XII (2017-18)

Mathematics

[SECTION A]

1. Here two matrices are equal so comparing the corresponding elements

$x + y = 6$ ...(1)

$5 + z = 5 \Rightarrow z = 0$

and $x.y = 8$ ...(2)

Solving (i) and (ii) we get : $x = 4$ or 2 and $y = 2$ or 4

2. Given that $f(x) = 8x^3$ and $g(x) = x^{1/3}$

Now $gof = g(f(x)) = g(8x^3) = (8x^3)^{1/3} = 2x$

and $fog = f(g(x)) = f\left(x^{1/3}\right) = 8\left(x^{1/3}\right)^3 = 8x$

3. $\int (ax+b)^2 dx$

$$= \frac{(ax+b)^{2+1}}{2+1} \times \frac{1}{a} \qquad \left\{\because \int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{n+1} \times \frac{1}{a}\right\}$$

$$= \frac{1}{3a}(ax+b)^3 + C$$

4. Here line makes 90°, 135°, 45° with *x*, *y* and *z* axes respectively. So $l = \cos 90°$, $m = \cos 135°$, $n = \cos 45°$,

$l = 0, m = \frac{-1}{\sqrt{2}}, n = \frac{1}{\sqrt{2}}$

[SECTION B]

5. We have $(X+Y) + (X-Y) = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$

or $(X+X) + (Y-Y) = \begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix} \Rightarrow 2X = \begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix}$ or $X = \frac{1}{2}\begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}$

Also $(X+Y) - (X-Y) = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$

or $(X-X) + (Y+Y) = \begin{bmatrix} 5-3 & 2-6 \\ 0 & 9+1 \end{bmatrix} \Rightarrow 2Y = \begin{bmatrix} 2 & -4 \\ 0 & 10 \end{bmatrix}$ or $Y = \frac{1}{2}\begin{bmatrix} 2 & -4 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}$

6. $f(x) = 2x^2 - 3x$

Differentiating w.r.t. *x*,

$f'(x) = 4x - 3$

(a) For strictly increasing

$f'(x) > 0$

$\Rightarrow \quad 4x - 3 > 0 \quad \Rightarrow \quad x > \frac{3}{4}$ or $x \in \left(\frac{3}{4}, \infty\right)$

(b) For strictly decreasing

$f'(x) < 0$

$\Rightarrow \quad 4x - 3 < 0 \quad \Rightarrow \quad x < \frac{3}{4}$ or $x \in \left(-\infty, \frac{3}{4}\right)$

7. Differentiating $x^2 = 4y$ with respect to *x*, we get :

$$\frac{dy}{dx} = \frac{x}{2}$$

Let (*h*, *k*) be the coordinates of the foot of the normal to the curve $x^2 = 4y$. Now, slope of the tangent at (h, k) is given by $\left.\frac{dy}{dx}\right]_{(h,k)} = \frac{h}{2}$

Hence, slope of the normal at (*h*, *k*) = $\frac{-2}{h}$

Therefore, the equation of normal at (*h*, *k*) is $y - k = \frac{-2}{h}(x - h)$ …(1)

Since it passes through the point (1, 2), we have

$2 - k = \frac{-2}{h}(1 - h)$ or $k = 2 + \frac{2}{h}(1 - h)$ …(2)

Since (*h*, *k*) lies on the curve $x^2 = 4y$, we have

$h^2 = 4k$ …(3)

Form (2) and (3), we have h = 2 and k = 1. Substituting the values of *h* and *k* in (1), we get the required equation of normal as $y - 1 = \frac{-2}{2}(x - 2)$ or $x + y = 3$

8. If the profit function is $p(x) = 41 - 72x - 18x^2$

So differentiating w.r.t. *x*

we get : $p'(x) = -72 - 36x$

For maxima or minima, putting $p'(x) = 0 \quad \Rightarrow \quad -72 - 36x = 0 \quad \Rightarrow \quad x = -2$

Differentiating again $p''(x) = -36$

i.e. $p''(x) < 0$

So *p*(*x*) is maximum at *x* = –2

and maximum profit is $p(-2) = 41 - 72(-2) - 18(-2)^2 = 113$

So the company can make a maximum profit of Rs. 113.

9. Comparing the given equation with the standard form $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$

We observe that $x_1 = -3, y_1 = 5, z_1 = -6; a = 2, b = 4, c = 2$

Thus, the required line passes through the point (–3, 5, –6) and is parallel to the vector $2\hat{i} + 4\hat{j} + 2\hat{k}$. Let $\vec{r}$ be the position vector of any point on the line, then the vector equation of the line is given by $\vec{r} = \left(-3\hat{i} + 5\hat{j} - 6\hat{k}\right) + \lambda\left(2\hat{i} + 4\hat{j} + 2\hat{k}\right)$.

10. **(i)** $\because$ $A \subseteq B$

$\Rightarrow$ $A \cap B = A$ and $P(A \cap B) = P(A)$

So $P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)} = \frac{P(A)}{P(A)} = 1$

(ii) $\because$ $A \cap B = \phi$

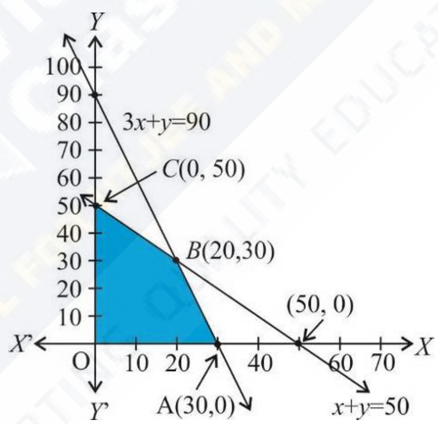
$\Rightarrow$ $P(A \cap B) = 0$

So $P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)} = \frac{0}{P(A)} = 0 \ (\because P(A) \neq 0)$

11. The shaded region in fig. is the feasible region determined by the system of constraints (2) to (4). We observe that the feasible region *OABC* is bounded. So, we now use corner point method to determine the maximum value of *Z*.

The coordinates of the corner points *O*, *A*, *B* and *C*, are (0, 0), (30, 0), (20, 30) and (0, 50) respectively. Now we evaluate *Z* at each corner point.

Corner Point	Corresponding value of Z	
(0, 0)	0	
(30, 0)	120 $\leftarrow$	Maximum
(20, 30)	110	
(0, 50)	50	



Hence, maximum value of Z is 120 at the point (30, 0).

12. We have $x^2 - 6x + 13 = x^2 - 6x + 3^2 - 3^2 + 13 = (x-3)^2 + 4$

So, $\int \frac{dx}{x^2 - 6x + 13} = \int \frac{1}{(x-3)^2 + 2^2} dx$

Let $x - 3 = t$. then $dx = dt$

Therefore, $\int \frac{dx}{x^2 - 6x + 13} = \int \frac{dt}{t^2 + 2^2} = \frac{1}{2} \tan^{-1} \frac{t}{2} + C = \frac{1}{2} \tan^{-1} \frac{x-3}{2} + C$

[SECTION C]

13. Given that $0 < x < 1$

and $\sqrt{1+x^2}\left[\left\{x\cos\left(\cot^{-1}x\right)+\sin\left(\cot^{-1}x\right)\right\}^2-1\right]^{1/2}=2x$

$$\Rightarrow \quad \sqrt{1+x^2}\left[\left\{x\cos\left(\cos^{-1}\frac{x}{\sqrt{1+x^2}}\right)+\sin\left(\sin^{-1}\frac{1}{\sqrt{1+x^2}}\right)\right\}^2-1\right]^{1/2}=2x$$

$$\Rightarrow \quad \sqrt{1+x^2}\left[\left\{\frac{x^2}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+x^2}}\right\}^2-1\right]^{1/2}=2x \quad \Rightarrow \quad \sqrt{1+x^2}\left[\left\{\frac{x^2+1}{\sqrt{1+x^2}}\right\}^2-1\right]^{1/2}=2x$$

$$\Rightarrow \quad \sqrt{1+x^2}\left[1+x^2-1\right]^{1/2}=2x \qquad \Rightarrow \quad \sqrt{1+x^2}.|x|=2x$$

$$\Rightarrow \quad \sqrt{1+x^2}.x=2x\left(\because\ 0<x<1\right) \qquad \Rightarrow \quad \sqrt{1+x^2}=2$$

$$\Rightarrow \quad 1+x^2=4 \qquad \Rightarrow \quad x=\pm\sqrt{3} \qquad \text{So solution is } x=\sqrt{3}\left(\because 0<x<1\right)$$

14. Given that $A=\begin{bmatrix}0 & -\tan\frac{\alpha}{2}\\ \tan\frac{\alpha}{2} & 0\end{bmatrix}$ and $I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

So, $\quad \text{L.H.S} = \text{I + A} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\begin{bmatrix}0 & -\tan\frac{\alpha}{2}\\ \tan\frac{\alpha}{2} & 0\end{bmatrix}=\begin{bmatrix}1 & -\tan\frac{\alpha}{2}\\ \tan\frac{\alpha}{2} & 1\end{bmatrix}$

and $\quad \text{R.H.S} = \left(I-A\right)\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}$

$$=\begin{bmatrix}1 & \tan\frac{\alpha}{2}\\ -\tan\frac{\alpha}{2} & 1\end{bmatrix}\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}=\begin{bmatrix}\cos\alpha+\sin\alpha.\tan\frac{\alpha}{2} & -\sin\alpha+\cos\alpha.\tan\frac{\alpha}{2}\\ -\cos\alpha.\tan\frac{\alpha}{2}+\sin\alpha & \sin\alpha.\tan\frac{\alpha}{2}+\cos\alpha\end{bmatrix}$$

Now putting $\sin\alpha=\dfrac{2\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$ and $\cos\alpha=\dfrac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}=\begin{bmatrix}1 & -\tan\frac{\alpha}{2}\\ \tan\frac{\alpha}{2} & 1\end{bmatrix}$

L.H.S = R.H.S
Hence proved.

OR

Applying operations $R_2 \to R_2-2R_1$ and $R_3 \to R_3-3R_1$ to the given determinant Δ, we have

$$\Delta=\begin{vmatrix}a & a+b & a+b+c\\ 0 & a & 2a+b\\ 0 & 3a & 7a+3b\end{vmatrix}$$

Now applying $R_3 \to R_3-3R_2$, we get : $\Delta=\begin{vmatrix}a & a+b & a+b+c\\ 0 & a & 2a+b\\ 0 & 0 & a\end{vmatrix}$

Expanding along C_1, we obtain $\Delta=a\begin{vmatrix}a & 2a+b\\ 0 & a\end{vmatrix}+0+0=a\left(a^2-0\right)=a\left(a^2\right)=a^3$

15. $y=\left(\tan^{-1}x\right)^2$ ...(1)

Differentiating w.r.t. x

$$\frac{dy}{dx}=2\left(\tan^{-1}x\right).\frac{1}{1+x^2}$$

$$\left(1+x^2\right)^2\left(\frac{dy}{dx}\right)^2=4\left(\tan^{-1}x\right)^2$$

$$\left(1+x^2\right)^2\left(\frac{dy}{dx}\right)^2=4y \qquad \text{(from equation (1))}$$

Differentiating again w.r.t. to x

$$2\left(1+x^2\right)\times 2x\left(\frac{dy}{dx}\right)^2+\left(1+x^2\right)^2\times 2\frac{dy}{dx}\times\frac{d^2y}{dx^2}=4\frac{dy}{dx}$$

$$2\frac{dy}{dx}\left[2\left(1+x^2\right)\left(\frac{dy}{dx}\right)x+\left(1+x^2\right)^2\frac{d^2y}{dx^2}\right]=4\frac{dy}{dx}$$

$$\left(1+x^2\right)^2\frac{d^2y}{dx^2}+2x\left(1+x^2\right)\frac{dy}{dx}=2 \qquad \left[\frac{dy}{dx}\neq 0\right]$$

OR

Differentiate the given function f w.r.t. x.

$$f(x)=(\log x)^x+x^{\log x}$$

OR

$$f(x)=(\ln x)^x+(x)^{\ln x}$$

Let $u=(\ln x)^x$ and $v=(x)^{\ln x}$

Taking log on both sides

$\ln u=x.\ln(\ln x)$

Differentiating w.r.t. x

$$\frac{1}{u}\times\frac{du}{dx}=x.\frac{1}{\ln x}.\frac{1}{x}+\ln(\ln x)$$

$$\frac{du}{dx}=u\left[\frac{1}{\ln x}+\ln(\ln x)\right] \qquad .....(1)$$

and $v=(x)^{\ln x}$

Taking log on both sides

$$\ln v=\ln x.\ln x=(\ln x)^2$$

Differentiating w.r.t x

$$\frac{1}{v}\frac{dv}{dx}=2\ln x\times\frac{1}{x} \quad\Rightarrow\quad \frac{dv}{dx}=(x)^{\ln x}\times\frac{2\ln x}{x}$$

$$\frac{dv}{dx}=2(\ln x).(x)^{\ln x-1} \qquad ...(2)$$

So $f(x)=u+v$

Differentiating w.r.t. x

$$f'(x)=\frac{du}{dx}+\frac{dv}{dx}$$

$$\Rightarrow \quad f'(x)=(\ln x)^x\left[\frac{1}{\ln x}+\ln(\ln x)\right]+2(\ln x)(x)^{\ln x-1} \qquad \text{(From (1) and (2))}$$

16. Let $y = \sin\phi$

Then $dy = \cos\phi\, d\phi$

Therefore, $\int \frac{(3\sin\phi - 2)\cos\phi}{5 - \cos^2\phi - 4\sin\phi} d\phi = \int \frac{(3y-2)dy}{5 - (1 - y^2) - 4y}$

$$= \int \frac{3y-2}{y^2 - 4y + 4} dy = \int \frac{3y-2}{(y-2)^2} = I(say)$$

Now we write $\frac{3y-2}{(y-2)^2} = \frac{A}{y-2} + \frac{B}{(y-2)^2}$ [by Table 7.2 (2)]

Therefore $3y - 2 = A(y-2) + B$

Comparing the coefficients of y and constant term, we get A = 3 and B –2A = –2, which gives A = 3 and B = 4 Therefore, the required integral given by

$$I = \int \left[\frac{3}{y-2} + \frac{4}{(y-2)^2} \right] dy = 3\int \frac{dy}{y-2} + 4\int \frac{dy}{(y-2)^2} = 3\log_e |y-2| + 4\left(-\frac{1}{(y-2)}\right) + C$$

$= 3\log_e (2 - \sin\phi) + \frac{4}{2 - \sin\phi} + C$ (since, $2 - \sin\phi$ is always positive)

17. Let $I = \int_0^{\frac{\pi}{2}} \log\sin x dx$

$$I = \int_0^{\frac{\pi}{2}} \log\sin\left(\frac{\pi}{2} - x\right) dx = \int_0^{\frac{\pi}{2}} \log\cos x dx$$

Adding the two values of I, we get

$$2I = \int_0^{\frac{\pi}{2}} (\log\sin x + \log\cos x) dx$$

$= \int_0^{\frac{\pi}{2}} (\log\sin x\cos x + \log 2 - \log 2) dx$ (by adding and subtracting log2)

$$= \int_0^{\frac{\pi}{2}} \log\sin 2x dx - \int_0^{\frac{\pi}{2}} \log 2 dx$$

Put $2x = t$ in the first integral. Then $2dx = dt$, when $x = 0, t = 0$ and when $x = \frac{\pi}{2}$, $t = \pi$

Therefore $2I = \frac{1}{2}\int_0^{\pi} \log\sin t dt - \frac{\pi}{2}\log 2$

$= \frac{2}{2}\int_0^{\frac{\pi}{2}} \log\sin t dt - \frac{\pi}{2}\log 2$ [As $\sin(\pi - t) = \sin t$]

$= \int_0^{\frac{\pi}{2}} \log\sin x dx - \frac{\pi}{2}\log 2$ (by changing variable *t* to *x*)

$= I - \frac{\pi}{2}\log 2$

Hence $\int_0^{\frac{\pi}{2}} \log\sin x dx = \frac{-\pi}{2}\log 2.$

OR

Evaluate the definite integrals in $\int_1^4 [|x-1| + |x-2| + |x-3|] dx$

OR

$$\int_1^4 \left(|x-1|+|x-2|+|x-3|\right)dx$$

$$=\int_1^2 \left((x-1)-(x-2)-(x-3)\right)dx+\int_2^3 \left((x-1)+(x-2)-(x-3)\right)dx+\int_3^4 \left((x-1)+(x-2)+(x-3)\right)dx$$

$$=\int_1^2 (x-1-x+2-x+3)\,dx+\int_2^3 (x-1+x-2-x+3)\,dx+\int_3^4 (x-1+x-2+x-3)dx$$

$$=\int_1^2 (-x+4)\,dx+\int_2^3 (x)\,dx+\int_3^4 (3x-6)\,dx = \left[-\frac{x^2}{2}+4x\right]_1^2, +\left[\frac{x^2}{2}\right]_2^3+\left[\frac{3x^2}{2}-6x\right]_3^4 = \frac{19}{2}$$

18. **(a)** Note that if θ is the angle between AB and CD, then θ is also the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$.

Now $\overrightarrow{AB}$ = Position vector of B – position vector of A $=(2\hat{i}+5\hat{j})-(\hat{i}+\hat{j}+\hat{k})=\hat{i}+4\hat{j}-\hat{k}$

Therefore $\left|\overrightarrow{AB}\right|=\sqrt{(1)^2+(4)^2+(-1)^2}=3\sqrt{2}$

Similarly $\overrightarrow{CD}=-2\hat{i}-8\hat{j}+2\hat{k}$ and $\left|\overrightarrow{CD}\right|=6\sqrt{2}$

Thus $\cos\theta=\dfrac{\overrightarrow{AB}.\overrightarrow{CD}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{CD}\right|}=\dfrac{1(-2)+4(-8)+(-1)(2)}{(3\sqrt{2})(6\sqrt{2})}=\dfrac{-36}{36}=-1$

Since $0\le\theta\le\pi$, it follows that $\theta=\pi$. This shows that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear

Alternatively, $\overrightarrow{AB}=-\dfrac{1}{2}\overrightarrow{CD}$ which implies that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear vectors.

(b) Since $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, we have $\vec{a}.\left(\vec{a}+\vec{b}+\vec{c}\right)=0$

Or $\quad \vec{a}.\vec{a}+\vec{a}.\vec{b}+\vec{a}.\vec{c}=\vec{0}$

Therefore $\quad \vec{a}.\vec{b}+\vec{a}.\vec{c}=-\left|\vec{a}\right|^2=-1$ …(1) Again $\vec{b}.\left(\vec{a}+\vec{b}+\vec{c}\right)=0$

Or $\quad \vec{a}.\vec{b}+\vec{b}.\vec{c}=-\left|\vec{b}\right|^2=-16$ …(2)

Similarly $\vec{a}.\vec{c}+\vec{b}.\vec{c}=-4$ …(3)

Adding (1), (2) and (3), we have $2\left(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{a}.\vec{c}\right)=-21$

Or $\quad 2\mu=-21, i.e., \mu=\dfrac{-21}{2}$

19. The given function is $y=e^{ax}\left[c_1\cos bx+c_2\sin bx\right]$ …(1)

Differentiating both sides of equation (1) with respect to x, we get :

$$\frac{dy}{dx}=e^{ax}\left[-bc_1\sin bx+bc_2\cos bx\right]+\left[c_1\cos bx+c_2\sin bx\right]e^{ax}.a$$

Or $\quad \dfrac{dy}{dx}=e^{ax}\left[(bc_2+ac_1)\cos bx+(ac_2-bc_1)\sin bx\right]$ …(2)

Differentiating both sides of equation (2) with respect to x, we get :

$$\frac{d^2y}{dx^2}=e^{ax}\left[(bc_2+ac_1)(-b\sin bx)+(ac_2-bc_1)(b\cos bx)\right]+\left[(bc_2+ac_1)\cos bx+(ac_2-bc_1)\sin bx\right]e^{ax}.a$$

$$=e^{ax}\left[\left(a^2c_2-2abc_1-b^2c_2\right)\sin bx+\left(a^2c_1+2abc_2-b^2c_1\right)\cos bx\right]$$

Substituting the values of $\frac{d^2y}{dx^2}, \frac{dy}{dx}$ and y in the given differential equation, we get :

$$\text{L.H.S.} = e^{ax}\left[a^2c_2 - 2abc_1 - b^2c_2)\sin bx + \left(a^2c_1 + 2abc_2 - b^2c_1\right)\cos bx\right]$$

$$-2ae^{ax}\left[\left(bc_2 + ac_1\right)\cos bx + \left(ac_2 - bc_1\right)\sin bx\right] + \left(a^2 + b^2\right)e^{ax}\left(c_1\cos bx + c_2\sin bx\right)$$

$$= e^{ax}\begin{bmatrix}\left(a^2c_2 - 2abc_1 - b^2c_2 - 2a^2c_2 + 2abc_1 + a^2c_2 + b^2c_2\right)\sin bx \\ +\left(a^2c_1 + 2abc_2 - b^2c_1 - 2abc_2 - 2a^2c_1 + a^2c_1 + b^2c_1\right)\cos bx\end{bmatrix}$$

$$= e^{ax}\left[0 \times \sin bx + 0\cos bx\right] = e^{ax} \times 0 = 0 = R.H.S.$$

20. Given that $|\vec{a}| = |\vec{b}| = |\vec{c}| = k\,(say)$

and $\quad \vec{a}.\vec{b} = \vec{b}.\vec{c} = \vec{c}.\vec{a} = 0$

Let $\quad \vec{d} = \vec{a} + \vec{b} + \vec{c}$

Now $\quad \left|\vec{d}\right|^2 = \left|\vec{a} + \vec{b} + \vec{c}\right|^2$

$$= |\vec{a}|^2 + \left|\vec{b}\right|^2 + |\vec{c}|^2 + 2\left(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}\right) = k^2 + k^2 + k^2$$

$$\left|\vec{d}\right|^2 = 3k^2 \Rightarrow \left|\vec{d}\right| = \sqrt{3}k$$

Let θ_1 be the angle between $\vec{a}$ and $\vec{d}$ so $\vec{a}.\vec{d} = |\vec{a}|\left|\vec{d}\right|\cos\theta_1$

$$\vec{a}.\left(\vec{a} + \vec{b} + \vec{c}\right) = k.\sqrt{3}k\cos\theta_1$$

$\Rightarrow \quad |\vec{a}|^2 + \vec{a}.\vec{b} + \vec{a}.\vec{c} = \sqrt{3}k^2\cos\theta_1 \quad \Rightarrow \quad \cos\theta_1 = \frac{k^2}{\sqrt{3}k^2} = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad \theta_1 = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

Similarly the angle between $\vec{b},\vec{d}$ and $\vec{c},\vec{d}$ is $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

So we can say that $\vec{d}$ is equally inclined to the vectors $\vec{a},\vec{b}$ and $\vec{c}$, and the angle is $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

21. The Probability distribution of X is

X	0	1	2	3	4
P(X)	0.1	k	2k	2k	k

(a) We know that $\sum_{i=1}^{n} p_i = 1$

Therefore $0.1 + k + 2k + 2k + k = 1$

i.e. $k = 0.15$

(b) P(you study at least two hours) $= P(X \geq 2)$

$$= P(X = 2) + P(X = 3) + P(X = 4)$$

$$= 2k + 2k + k = 5k = 5 \times 0.15 = 0.75$$

P(you study exactly two hours) $= P(X = 2)$

$$= 2k = 2 \times 0.15 = 0.3$$

P (you study at most two hours) $= P(X \leq 2)$

$$= P(X = 0) + P(X = 1) + P(X = 2)$$

$$= 0.1 + k + 2k = 0.1 + 3k = 0.1 + 3 \times 0.15$$

$$= 0.55$$

22. Let E be the event that the doctor visits the patient late and let T_1, T_2, T_3, T_4 be the events that the doctor comes by train, bus, scooter and other means of transport respectively.

Then $P(T_1) = \frac{3}{10}, P(T_2) = \frac{1}{5}, P(T_3) = \frac{1}{10}$ and $P(T_4) = \frac{2}{5}$ (given)

$P(E/T_1) =$ Probability that the doctor arriving late comes by train $= \frac{1}{4}$

Similarly, $P(E/T_2) = \frac{1}{3}, P(E|T_3) = \frac{1}{12}$ and $P(E|T_4) = 0,$ since he is not late if he comes by other means of transport.

Therefore, by Bayes' Theorem, we have

$P(T_1|E) =$ Probability that the doctor arriving late comes by train

$$= \frac{P(T_1)P(E|T_1)}{P(T_1)P(E|T_1) + P(T_2)P(E|T_2) + P(T_3)P(E|T_3) + P(T_4)P(E|T_4)}$$

$$= \frac{\frac{3}{10} \times \frac{1}{4}}{\frac{3}{10} \times \frac{1}{4} + \frac{1}{5} \times \frac{1}{3} + \frac{1}{10} \times \frac{1}{12} + \frac{2}{5} \times 0} = \frac{3}{40} \times \frac{120}{18} = \frac{1}{2}$$

Hence, the required probability is $\frac{1}{2}$.

23. Let x and y be the number of items M and N respectively.

Total profit on the production = Rs (600 x + 400 y)

Mathematical formulation of the given problem is as follows.

Maximise $Z = 600x + 400y$

Subject to the constraints

$x + 2y \le 12$ (constraint on Machine I) ...(1)

$2x + y \le 12$ (constraint on Machine II) …(2)

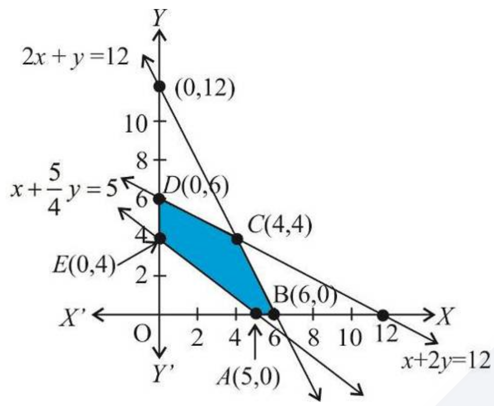
$x + \frac{5}{4}y \ge 5$ (constraint on Machine III) …(3)

$x \ge 0, y \ge 0$ …(4)

Let us draw the graph of constraint (1) to (4). *ABCDE* is the feasible region (shaded) as shown in figure. determined by the constraint (1) to (4). Observe that the feasible region is bounded, coordinates of the corner points A, B, C, D and E are (5, 0) (6, 0), (4, 4), (0, 6) and (0, 4) respectively.

Let us evaluate $Z = 600x + 400y$ at these corner points.

Corner Point	Z = 600*x* + 400*y*
(5, 0)	3000
(6, 0)	3600
(4, 4)	4000 ← maximum
(0, 6)	2400
(0, 4)	1600



We see that the point (4, 4) is giving the maximum value of Z. Hence, the manufacturer has to produce 4 units of each item to get the maximum profit of Rs 4000.

[SECTION D]

24. Let first, second and third numbers be denoted by *x*, *y* and *z*, respectively. Then, according to given conditions, We have

$x+y+z=6$

$y+3z=11$

$x+z=2y$ or $x-2y+z=0$

This system can be written as *A X* = *B*, where $A=\begin{bmatrix}1&1&1\\0&1&3\\1&-2&1\end{bmatrix}, X=\begin{bmatrix}x\\y\\z\end{bmatrix}$ and $B=\begin{bmatrix}6\\11\\0\end{bmatrix}$

Here $|A|=1(1+6)-(0-3)+(0-1)=9\neq 0.$ Now we find adj *A*

$a_{11}=1(1+6)=7,\quad a_{12}=-(0-3)=3,\quad a_{13}=-1$

$a_{21}=-1(1+2)=-3,\quad a_{22}=0,\quad a_{23}=-(-2-1)=3$

$a_{31}=(3-1)=2,\quad a_{32}=-(3-0)=-3,\quad a_{33}=(1-0)=1$

Hence $adj\,A=\begin{bmatrix}7&-3&2\\3&0&-3\\-1&3&1\end{bmatrix}$ Thus $A^{-1}=\frac{1}{|A|}adj(A)=\frac{1}{9}\begin{bmatrix}7&-3&2\\3&0&-3\\-1&3&1\end{bmatrix}$

Since $X=A^{-1}B$

$X=\frac{1}{9}\begin{bmatrix}7&-3&2\\3&0&-3\\-1&3&1\end{bmatrix}\begin{bmatrix}6\\11\\0\end{bmatrix}$ Or $\begin{bmatrix}x\\y\\z\end{bmatrix}=\frac{1}{9}\begin{bmatrix}42-33+0\\18+0+0\\-6+33+0\end{bmatrix}=\frac{1}{9}\begin{bmatrix}9\\18\\27\end{bmatrix}=\begin{bmatrix}1\\2\\3\end{bmatrix}$

Thus $x=1, y=2, z=3$

25. It is given that * : $R\times R\to R$ is defined as $a*b=|a-b|\forall a,b\in R$

So $a*b=|a-b|=|b-a|=b*a\forall a,b\in R$

So $a*b=b*a\ \forall a,b\in R$

So * is commutative operation

Now to check associativity

$a*(b*c)=a*|b-c|=|a-|b-c||\forall a,b,c\in R$ and $(a*b)*c=|a-b|*c=||a-b|-c|\forall a,b,c\in R.$

Clearly $a*(b*c)\neq(a*b)*c$

So * is not associative

Again $o : R \times R \to R$ is defined as a $aob = a \forall a, b \in R$

$aob = a$ and $boa = b \forall a, b \in R$

So operation 'o' is not commutative.

and $ao(boc) = aob = a \forall a, b, c \in R$

and $(aob)oc = aoc = a \forall\ a, b, c \in R$

So operation 'o' is associative.

Further $a*(boc) = a*b\left[\because boc = b\right] = |a-b|$ and $(a*b)o(a*c) = |a-b|o|a-c| = |a-b|$

So $a*(boc) = (a*b)0(a*c) \forall a, b, c \in R$

Now to check 'O' is distributive over '*'

$ao(b*c) = ao|b-c| = a$

and $(aob)*(a0c) = a*a = |a-a| = 0 \forall a, b, c \in R$

here $ao(b*c) \neq (aob)*(aoc)$

So 'o' is not distributive over *R*.

OR

$f : R^+ \to (-5, \infty)$

$f(x) = 9x^2 + 6x - 5 = y$

$\Rightarrow \quad 9x^2 + 6x - 5 - y = 0 \quad \Rightarrow \quad x = \dfrac{-6 \pm \sqrt{36 - 4 \times 9 \times (-5-y)}}{2 \times 9}$

$\Rightarrow \quad x = \dfrac{-6 \pm 6\sqrt{1+5+y}}{18} = \dfrac{-1 \pm \sqrt{y+6}}{3}$

$\because \quad x \in R+$ so neglecting minus sign

$\Rightarrow \quad x = \dfrac{-1+\sqrt{y+6}}{3} = f^{-1}(y)$

Hence $f^{-1}(y) = \dfrac{\sqrt{y+6}-1}{3}$

(i) $\quad f^{-1}(3) = \dfrac{\sqrt{3+6}-1}{3} = \dfrac{3-1}{3} = \dfrac{2}{3}$

(ii) $\quad f^{-1}(y) = \dfrac{4}{3}$

$\Rightarrow \quad \dfrac{\sqrt{y+6}-1}{3} = \dfrac{4}{3}$

$\Rightarrow \quad \sqrt{y+6} = 5$

$\Rightarrow \quad y = 19$

26. It is given that height of cone is h and semi vertical angle is α. Let r be the radius and h' be the height of cylinder.

Now in ΔAEH, $\tan\alpha = \dfrac{EH}{AH} = \dfrac{r}{AH}$

$\Rightarrow \quad AH = r\cot\alpha$, so $AI = AH + HI$

$\Rightarrow \quad h = r\cot\alpha + h'$

$\Rightarrow \quad h' = h - r\cot\alpha \quad$ …..(1)

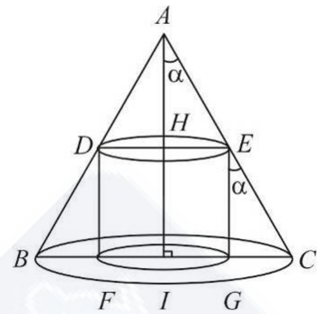
Now let V be the volume of cylinder

So $V = \pi r^2 h' = \pi r^2 (h - r\cot\alpha)$ (from equation (1))

$$V = \pi h r^2 - \pi r^3 \cot\alpha$$

Differentiating w.r.to r

$$\frac{dV}{dr} = 2\pi h r - 3\pi r^2 \cot\alpha \quad \ldots(2)$$



For maxima and minima, putting $\frac{dV}{dr} = 0$

$$2\pi h r - 3\pi r^2 \cot\alpha = 0$$

$\Rightarrow$ $2h = 3r\cot\alpha$ $\Rightarrow$ $r = \frac{2h}{3}\tan\alpha$

and $h' = h - \frac{2h}{3}\tan\alpha.\cot\alpha$

$\Rightarrow$ $h' = \frac{1}{3}h$ which is we have to prove.

Again differentiating equation (2) w.r.t. r

$$\frac{d^2V}{dr^2} = 2\pi h - 6\pi r\cot\alpha = 2\pi h - 6\pi.\frac{2h}{3}\tan\alpha.\cot\alpha = -2\pi h$$

$\Rightarrow$ $\frac{d^2V}{dr^2} < 0$

So volume of cylinder is maxima.

and maximum volume = $\pi r^2 h' = \pi \times \frac{4}{9}h^2 \tan^2\alpha \times \frac{1}{3}h = \frac{4}{27}\pi h^3 \tan^2\alpha.$ (Hence proved)

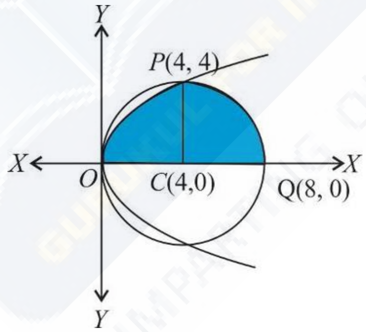
27. The given equation of the circle $x^2 + y^2 = 8x$ can be expressed as $(x-4)^2 + y^2 = 16$. Thus, the center of the circle is (4, 0) and radius is 4. Its intersection with the parabola $y^2 = 4x$ gives

$$x^2 + 4x = 8x$$

or $x^2 - 4x = 0$ or $x(x-4) = 0$ or $x = 0, x = 4$

Thus the points of intersection of these two curves are O(0, 0) and P(4, 4) above the x-axis.

From the fig the required area of the region *OPQCO* included between these two curves above x-axis is



= (area of the region *OCPO*) + (area of the region *PCQP*)

$$= \int_0^4 y\,dx + \int_4^8 y\,dx = 2\int_0^4 \sqrt{x}\,dx + \int_4^8 \sqrt{4^2 - (x-4)^2}\,dx = 2 \times \frac{2}{3}\left[x^{3/2}\right]_0^4 + \int_0^4 \sqrt{4^2 - t^2}\,dt \text{ where } x - 4 = t$$

$$= \frac{32}{3} + \left[\frac{t}{2}\sqrt{4^2 - t^2} + \frac{1}{2} \times 4^2 \times \sin^{-1}\frac{t}{4}\right]_0^4$$

$$= \frac{32}{3} + \left[\frac{4}{2} \times 0 + \frac{1}{2} \times 4^2 \times \sin^{-1} 1\right] = \frac{32}{3} + \left[0 + 8 \times \frac{\pi}{2}\right] = \frac{32}{3} + 4\pi = \frac{4}{3}(8 + 3\pi)$$

OR

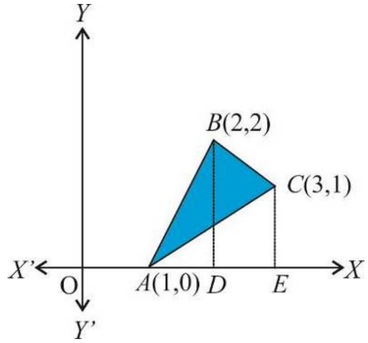
Using integration find the area of region bounded by the triangle whose vertices are (1, 0), (2, 2) and (3, 1).

OR

Let A (1, 0), B (2, 2) and C (3, 1) be the vertices of a triangle *ABC*.

Area of ΔABC = Area of ΔABD + area of trapezium BDEC – Area of ΔAEC

Now equation of the sides *AB*, *BC* and *CA* are given by



$y = 2(x-1), y = 4-x, y = \frac{1}{2}(x-1)$, respectively

Hence, area of $\Delta ABC = \int_1^2 2(x-1)dx + \int_2^3 (4-x)\,dx - \int_1^3 \frac{x-1}{2}dx$

$$= 2\left[\frac{x^2}{2} - x\right]_1^2 + \left[4x - \frac{x^2}{2}\right]_2^3 - \frac{1}{2}\left[\frac{x^2}{2} - x\right]_1^3$$

$$= 2\left[\left(\frac{2^2}{2} - 2\right) - \left(\frac{1}{2} - 1\right)\right] + \left[\left(4\times 3 - \frac{3^2}{2}\right) - \left(4\times 2 - \frac{2^2}{2}\right)\right] - \frac{1}{2}\left[\left(\frac{3^2}{2} - 3\right) - \left(\frac{1}{2} - 1\right)\right] = \frac{3}{2}$$

28. The given differential equation can be written as

$$\frac{dx}{dy} = \frac{2xe^{\frac{x}{y}} - y}{2ye^{\frac{x}{y}}} \qquad \ldots(1)$$

Let $F(x, y) = \dfrac{2xe^{\frac{x}{y}} - y}{2ye^{\frac{x}{y}}}$

Then $F(\lambda x, \lambda y) = \dfrac{\lambda\left(2xe^{\frac{x}{y}} - y\right)}{\lambda\left(2ye^{\frac{x}{y}}\right)} = \lambda^0\left[F(x, y)\right]$

Thus, F(x, y) is a homogeneous function of degree zero. Therefore, the given differential equation is a homogeneous differential equation.

To solve it, we make the substitution

$x = vy \qquad \ldots(2)$

Differentiating equation (2) with respect to *y*, we get :

$$\frac{dx}{dy} = v + y\frac{dv}{dy}$$

Substituting the value of x and $\frac{dx}{dy}$ in equation (1), we get :

$$v+y\frac{dv}{dy}=\frac{2ve^{v}-1}{2e^{v}} \quad \text{or} \quad y\frac{dv}{dy}=\frac{2ve^{v}-1}{2e^{v}}-v \quad \text{or} \quad y\frac{dv}{dy}=\frac{-1}{2e^{v}}$$

or $\quad 2e^{v}dv=\frac{-dy}{y} \quad$ or $\quad \int 2e^{v}.dv=-\int\frac{dy}{y} \quad$ or $\quad 2e^{v}=-\ln|y|+C$

and replacing v by $\frac{x}{y}$, we get :

$$2e^{\frac{x}{y}}+\ln|y|=C \qquad \ldots(3)$$

Substituting $x=0$ and $y=1$ in equation (3), we get :

$$2e^{0}+\ln|1|=C \Rightarrow C=2$$

Substituting the value of C in equation (3), we get :

$$2e^{\frac{x}{y}}+\ln|y|=2$$

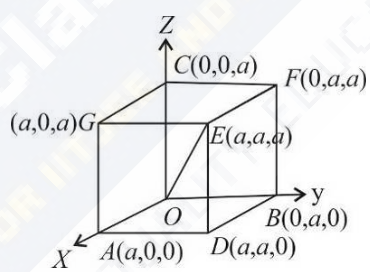
Which is the particular solution of the given differential equation.

29. A cube is a rectangular parallelepiped having equal length, breadth and height. Let $OADBFEGC$ be the cube with each side of length a unit. The four diagonals are *OE, AF, BG* and *CD.*

The direction cosines of the diagonal OE which is the line joining two points O and E are

$$\frac{a-0}{\sqrt{a^2+a^2+a^2}},\frac{a-0}{\sqrt{a^2+a^2+a^2}},\frac{a-0}{\sqrt{a^2+a^2+a^2}}$$

i.e. $\quad \frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}$



Similarly, the direction cosines of AF, BG and CD are $\frac{-1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}};\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}}$, respectively

Let l,m,n be the direction cosines of the given line which makes angles $\alpha,\beta,\gamma,\delta$ with OE,AF,BG,CD, respectively. Then :

$$\cos\alpha=\frac{1}{\sqrt{3}}(l+m+n);\cos\beta=\frac{1}{\sqrt{3}}(-l+m+n);\ \cos\gamma=\frac{1}{\sqrt{3}}(l-m+n);\cos\delta=\frac{1}{\sqrt{3}}(l+m-n)$$

Squaring and adding, we get :

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta$$

$$=\frac{1}{3}\left[(l+m+n)^2+(-l+m+n)^2+(l-m+n)^2+(l+m-n)^2\right]$$

$$=\frac{1}{3}\left[4\left(l^2+m^2+n^2\right)\right]=\frac{4}{3} \text{ (as } l^2+m^2+n^2=1)$$

OR

Find the equation of the plane which contains the line of intersection of the planes $\vec{r}.\left(\hat{i}+2\hat{j}+3\hat{k}\right)-4=0, \vec{r}.\left(2\hat{i}+\hat{j}-\hat{k}\right)+5=0$ and which is perpendicular to the plane $\vec{r}.\left(5\hat{i}+3\hat{j}-6\hat{k}\right)+8=0$ .Find whether the plane thus obtained contains the line $6x-12=-6y-6=25z-10$.

OR

We know that equation of any plane which pass through the line of intersection of planes $\vec{r}.\vec{n}_1=d_1$, and $\vec{r}.\vec{n}_2=d_2$ is $\vec{r}.(\vec{n}_1+\lambda\vec{n}_2)=d_1+\lambda d_2$ here $\vec{n}_1=\hat{i}+2\hat{j}+3\hat{k}$, $\vec{n}_2=2\hat{i}+\hat{j}-\hat{k}, d_1=4, d_2=-5$

So equation of required plane is

$$\vec{r}.\left((1+2\lambda)\hat{i}+(2+\lambda)\hat{j}+(3-\lambda)\hat{k}\right)=4-5\lambda \qquad ....(1)$$

But it is given that above plane is perpendicular to plane $\vec{r}.\left(5\hat{i}+3\hat{j}-6\hat{k}\right)+8=0$

So $\quad 5(1+2\lambda)+3(2+\lambda)-6(3-\lambda)=0 \qquad \Rightarrow \qquad \lambda=7/19$

Putting the value of λ in equation (1), we get required plane is $\vec{r}.\left[\left(1+\frac{14}{19}\right)\hat{i}+\left(2+\frac{7}{19}\right)\hat{j}+\left(3-\frac{7}{19}\right)\hat{k}\right]=4-\frac{35}{19}$

$\Rightarrow \qquad \vec{r}.\left(33\hat{i}+45\hat{j}+50\hat{k}\right)=41 \qquad$ or $\qquad 33x+45y+50z=41 \qquad ...(2)$

Now we check the given line $6x-12=-6y-6=25z-10$ lie on plane (2) or not.

So the line $6(x-2)=-6(y+1)=25\left(z-\frac{2}{5}\right)$

$$\Rightarrow \frac{x-2}{1/6}=\frac{y+1}{-1/6}=\frac{z-2/5}{1/25}$$

Passes through $\left(2,-1,\frac{2}{5}\right)$ whose d.r's are $\left(\frac{1}{6},\frac{-1}{6},\frac{1}{25}\right)$

Here $\quad 33\times\frac{1}{6}-45\times\frac{1}{6}+50\times\frac{1}{25}=\frac{11}{2}-\frac{15}{2}+\frac{4}{2}=0 \quad$ and $\quad 33\times2-45\times1+50\times\frac{2}{5}=66-45+20=41\neq0$

So the given line not lie on the plane 2.